BIFURCATION AND CHAOS OF SYNCHRONIZED STATES IN OSCILLATORS WITH HARD CHARACTERISTICS AND STATE COUPLING

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We investigate the phase transition between solutions with distinct symmetrical property observed in a system of coupled three-oscillators with hard characteristics and state coupling. By a symmetry-breaking bifurcation, a symmetrical in-phase solution bifurcates into synchronized modes with a partially in-phase solution and an almost in-phase solution. Moreover, by using the definition of symmetrical and asymmetrical three-phase solutions, we confirmed the existence of a stable symmetrical three-phase quasi-periodic solution and an asymmetrical three-phase chaotic solution in the coupled system.

Keywords: Coupled oscillator; bifurcation; symmetry; three-phase solution.

1. Introduction

Systems of coupled oscillators are widely used as models for biological rhythmic oscillations such as human circadian rhythms [Kronauer et al., 1982; Brown et al., 2000], finger movements [Hirao et al., 1996], animal locomotion [Collins & Stewart, 1993; Golubitsky & Stewart, 1999; Buono & Golubitsky, 2001], swarms of fireflies that flash in synchrony [Winfree, 1980; Kousaka et al., 1998], synchronous firing of cardiac pacemaker cells [Winfree, 1980; Sousa et al., 1994], neural networks [Skinner et al., 1994; Han et al., 1997; Medvedev & Kopell, 2000], and so on.

Using these coupled oscillator models, many investigators have studied the mechanism of generation of synchronous oscillation and phase transitions between distinct oscillatory modes. From the standpoint of bifurcation, the former and the latter correspond respectively to the Hopf bifurcation of an equilibrium point (or the tangent bifurcation of a periodic solution) and to the pitchfork bifurcation (or the period-doubling bifurcation) of a periodic solution. Using a group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning the bifurcations of equilibrium points and periodic solutions [Golubitsky et al., 1988].

Our research group has investigated a system of a small number of coupled oscillators, aiming to classify periodic solutions according to their symmetrical properties, and to clarify the phase transition between classified periodic solutions [Papy & Kawakami, 1995a, 1995b; Kitajima & Kawakami, 1998]. We consider that the case of a small number of oscillators is a prototype modeling for understanding the phenomena in the case of a large number of oscillators. Shiohama and Kawakami [1998] studied a system of coupled three oscillators through inductors in a ring. The ring
structure is one of the simplest cases of coupled oscillators [Friesen & Stent, 1977; Tsutsumi & Matsumoto, 1984; Ermentrout, 1985], and “three” is the simplest case of the ring structure. They confirmed four kinds of stable periodic solutions and also showed chaotic oscillations caused by successive period-doubling bifurcations.

We have also studied a system of coupled three-oscillators with hard characteristics and state coupling, to obtain more stable states. The single oscillator, called a hard oscillator, has a stable equilibrium point and a stable periodic solution. We have classified the periodic solutions into twelve kinds according to their symmetrical property, and confirmed nine kinds of stable periodic solutions [Yamakawa et al., 1999].

In this paper, we further investigate the phase transition between solutions with distinct symmetrical properties. Moreover, by using the definition of three-phase solutions, we confirmed the existence of a stable symmetrical three-phase quasi-periodic solution and an asymmetrical three-phase chaotic solution. To the best of our knowledge, there is no paper describing an $n$-phase ($n \geq 3$) chaotic solution based on the common mathematical definition. We consider that these results give useful information for the design of a coupled oscillator system.

2. Preliminaries

2.1. System equation

Consider a system of coupled three-oscillators in a ring, as shown in Fig. 1. After normalization of the state variables and parameters, we obtain the following circuit equations:

$$\begin{align*}
\frac{dx_i}{dt} &= -\alpha x_i - \beta x_i^3 - \gamma x_i^5 + \omega x_{i+1} \\
+ \omega_0(x_{i+5} - x_{i+8}) \\
\frac{dx_{i+1}}{dt} &= -\omega x_i - \sigma x_{i+1} \\
\frac{dx_{i+2}}{dt} &= -\sigma_0 x_{i+2} + \omega_0(x_{i+3} - x_{i+6}) \\
& (i = 1, 4, 7, x_{10} \equiv x_1, x_{12} \equiv x_3, x_{13} \equiv x_4, \\
x_{15} \equiv x_6),
\end{align*}$$

where

$$\begin{align*}
x_1 &= \sqrt{C}v_1, x_2 = \sqrt{L}i_1, x_3 = \sqrt{L_0}i_4, \\
x_4 &= \sqrt{C}v_2, x_5 = \sqrt{L}i_2, x_6 = \sqrt{L_0}i_5, \\
x_7 &= \sqrt{C}v_3, x_8 = \sqrt{L}i_3, x_9 = \sqrt{L_0}i_6,
\end{align*}$$

$$\alpha = \frac{a_1}{C}, \beta = \frac{a_3}{C^2}, \gamma = \frac{a_5}{C^3}, \sigma = \frac{r}{L}, \sigma_0 = \frac{R_0}{L_0},$$

$$\omega = \frac{1}{\sqrt{CL}}, \omega_0 = \frac{1}{\sqrt{C_0L_0}}.$$

We define the permutation matrix $P$, the flip matrix $Q$, and the inversion matrix $I_9$, as follows:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \otimes I_3,$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_9 = -I_9,$$

where $I_n$ denotes the $n \times n$ identity matrix. We also define a matrix group:

$$\Gamma = \{I_9, P, P^2, Q, QP, QP^2, T_9, T_9P, T_9P^2, \overline{T}_9Q, \overline{T}_9QP, \overline{T}_9QP^2 \}.$$  

Then, Eq. (1) is equivariant to a new system under the following coordinate transform:

$$x \mapsto gx, \forall g \in \Gamma.$$  

Note that $\Gamma$ has a dihedral subgroup:

$$D_3 = \{I_9, P, P^2, Q, QP, QP^2 \}.$$
and \( D_3 \) has a cyclic subgroup:
\[
C_3 = \{ I_9, P, P^2 \}
\]
(7)

Together with three conjugate subgroups:
\[
C_2 = \{ I_9, Q \}, \{ I_9, QP \}, \{ I_9, QP^2 \}
\]
(8)

2.2. Poincaré mapping

We assume that the solution of Eq. (1) is
\[
x(t) = \varphi(t, x_0, \lambda),
\]
(9)
where \( x_0 \) is an initial state:
\[
\varphi(0, x_0, \lambda) = x_0,
\]
(10)
and \( \lambda \) is a parameter. We define a Poincaré section \( \Pi \) for the trajectory \( \varphi(t, x_0, \lambda) \). Then, the Poincaré mapping \( T_\lambda \) is
\[
T_\lambda : \Pi \rightarrow \Pi; x_0 \mapsto \varphi(\tau, x_0, \lambda),
\]
(11)
where \( \tau \) is the time instant taken for the path of trajectory, which starts from \( x_0 \) and ends at the first return point to \( \Pi \).

2.3. Definition of symmetrical and asymmetrical three-phase solutions

We define a mapping \( T_P \) as
\[
T_P : \Pi \rightarrow \Pi; x_0 \mapsto P^{-1} \varphi(\tau, x_0, \lambda)
\]
(12)
and a set \( \Sigma(x_0) \) as
\[
\Sigma(x_0) = \{ T_P^k(x_0) \mid k \in N \}.
\]
(13)
If the set \( \Sigma(x_0) \) is invariant under the mapping \( T_P \):
\[
T_P(\Sigma(x_0)) = \Sigma(x_0)
\]
(14)
and is connected, then the solution \( \varphi(t, x_0, \lambda) \) is called an asymmetrical three-phase solution [Fiedler, 1988; Katsuta, 1995]. When the matrix \( P^T I_9 \) is used instead of \( P \), the solution is called a symmetrical three-phase solution, where “symmetrical” indicates that it is invariant under the inversion of state variables.

Definitions of other symmetrical solutions, observed in a system with the dihedral group \( D_3 \), can be found from [Katsuta, 1995; Shiohama & Kawakami 1998; Yamakawa et al., 1999].

3. Main Results

We fix the parameter values of Eq. (1) as
\[
\beta = -1.4, \quad \gamma = 0.4,
\]
\[
\sigma = 0.5, \quad \sigma_0 = 0.5.
\]
(15)

In the bifurcation diagrams shown in this section, the tangent, period-doubling, Neimark–Sacker bifurcation, and D-type of branching (pitchfork bifurcation) sets of an \( m \)-periodic solution, are indicated respectively by symbols \( G_j^m, I_j^m, N_j^m \) and \( D_j^m \), where \( j \) denotes the number that distinguishes different bifurcation sets of the same period. If \( m = 1 \), it will be omitted.

3.1. Transition between distinct oscillatory modes

Figure 2 shows a bifurcation diagram of periodic solutions on the parameter plane \((\omega_0, \alpha)\). In the region shaded by \( \square \) and \( \bigcirc \), respectively, a stable symmetrical in-phase solution and a symmetrical partially anti-phase solution exist.

By the degenerate symmetry-breaking pitchfork bifurcation (satisfying double pitchfork bifurcation conditions) \( D_1 \), the symmetrical in-phase solution bifurcates into a partially in-phase
3.2. Symmetrical three-phase quasi-periodic solution

A bifurcation diagram of a symmetrical three-phase periodic solution is shown in Fig. 4. In the shaded region we observed a stable symmetrical three-phase periodic solution. By decreasing the value of $\omega_0$ from this region, the Neimark–Sacker bifurcation $N_1$ occurs and the quasi-periodic solution (Fig. 5) is generated.

Figure 6 shows points of the Poincaré mapping $T_\lambda$ and (b) the mapping $T_{PI9}$ for the symmetrical three-phase quasi-periodic solution.

3.3. Asymmetrical three-phase quasi-periodic and chaotic solutions

Figure 7 shows a bifurcation diagram of the three-phase periodic solutions observed in Eq. (1) with $\omega = 0.5$. We observed a stable three-phase solution...
in the region shaded by \( N_2 \). By the Neimark–Sacker bifurcation \( N_2 \), the stable three-phase solution becomes unstable and the quasi-periodic solution (Fig. 8) is generated. This quasi-periodic solution does not satisfy the definition of symmetrical three-phase [Fig. 9(b)]; however, it satisfies the definition of asymmetrical three-phase [Fig. 9(c)]. Thus, we call it a three-phase quasi-periodic solution.

In Fig. 7, the tangent bifurcation sets \( G_2^1 \) and \( G_2^2 \) meet the Neimark–Sacker bifurcation set as cusp points at the points marked by open circles (the argument of the characteristic multipliers equals \( \pi \) radian). We only show the tangent bifurcation set of two-periodic solutions. However, the tangent bifurcation sets of various kinds of periodic solutions were also observed [Mihara & Kawakami, 1996; Kitajima & Kawakami, 1997], which are called the Arnold tongues [Arnold, 1983].

Figure 10 shows an enlarged bifurcation diagram of a part of Fig. 7. In the region surrounded by the tangent bifurcation sets \( G_2^1 \) and \( G_2^2 \), the period-doubling bifurcation set \( I^2 \) of a two-periodic solution, generated by the tangent bifurcations, exist. The intersecting points of the tangent bifurcation sets and the period-doubling set are codimension-two bifurcation points, called \( TP \) bifurcation point.

For the sake of completeness, we repeated the bifurcation diagram of a part of Fig. 7. The points marked by \( \circ \), \( \bullet \) and \( \triangledown \) indicate codimension-two bifurcation point, called \( TP \) bifurcation.
successive $TP$ bifurcations occur and chaotic solution is generated. The schematic diagram is shown in Fig. 11. We observed tree-like pattern of the tangent bifurcation sets, discussed in [Yoshinaga & Kawakami, 1989].

Figure 12 shows a one-parameter bifurcation diagram changing the parameter values along the line $l$ in Fig. 10. From this figure, we can confirm that the chaotic state is generated by successive period-doubling bifurcations. The chaotic solution, at the parameter value marked by $\textcircled{I}$, does not satisfy the definition of asymmetrical three-phase (see Fig. 13). However, after the symmetry-increasing crisis [Melbourne et al., 1993], it satisfies the definition of asymmetrical three-phase (see Fig. 14). Thus, we can observe the asymmetrical three-phase chaotic solution at the parameter value marked by $\textcircled{II}$ in Fig. 12.

4. Concluding Remarks

We have investigated the phase transition between the solutions with distinct symmetrical property observed in a system of coupled three-oscillators with hard characteristics and state coupling. By a symmetry-breaking bifurcation, a symmetrical in-phase solution bifurcates into synchronized modes, called a partially in-phase solution and an almost in-phase solution. We have summarized the results of possible phase transitions observed in the symmetrical system with respect to the dihedral group $D_3$ and the inversion of state variables.

Moreover, by using the definition of symmetrical and asymmetrical three-phase solutions, we confirmed the existence of a stable symmetrical three-phase quasi-periodic solution and an asymmetrical three-phase chaotic solution. We have clarified the bifurcation mechanism of generating
the three-phase chaotic solution: successive period-doubling bifurcations of phase locked state (asymmetrical solution) occur and the three-phase chaotic solution appears after symmetry-increasing bifurcation.

In coupled oscillator systems with symmetrical properties, it is an interesting open problem to find the universal symmetrical property of periodic solutions generated by phase locking phenomena.

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References


