BIFURCATION IN COUPLED BVP NEURONS WITH EXTERNAL IMPULSIVE FORCES

Hiroyuki Kitajima

Kagawa University,
Takamatsu, 761-0396 JAPAN
kitaji@eng.kagawa-u.ac.jp

Hiroshi Kawakami

The University of Tokushima,
Tokushima, 770-8506 JAPAN
kawakami@ee.tokushima-u.ac.jp

ABSTRACT

We investigate bifurcation and chaos observed in coupled BVP neurons with external impulsive forces. Although the single neuron without the external force has only one equilibrium point, combining these n ($n \geq 3$) neurons unidirectionally in a ring, n-phase periodic solutions are generated. Applying the impulsive forces, successive period-doubling bifurcations of the n-phase solution occur and chaotic states, namely n-phase chaos, appear. When $n = 2$ and 3, we find the parameter regions in which the switching phenomena of burst firing are observed. Moreover, the mechanisms of the switching phenomena are clarified by numerical bifurcation analysis.

1 Introduction

A neuron, or the fundamental element of the brain, generates various temporal patterns of spikes [1,2]. Among such firing patterns, synchronous firing of neurons in connection with neuronal signal processing has attracted much interest [3, 4]. It is important to consider global dynamics of networks composed of nonlinear neurons in order to clarify not only mechanisms of synchronous firing of neurons but also its functional roles [5].

In this paper we examine nonlinear dynamics and bifurcations of unidirectionally coupled Bonhöffer van der Pol (BVP or FitzHugh-Nagumo) neurons with external impulsive forces. The BVP or FitzHugh-Nagumo equation is a well-known neuron model representing the electrical behavior across a nerve membrane and has been widely studied [6–10].

The model equation analyzed in this paper is described as

$$\begin{aligned} \frac{dx_i}{dt} &= c\left(x_i - \frac{1}{3}x_i^3 + y_i\right) + h\sum_{k\in Z}\delta(t - k\tau) \\ &\quad + W_i \tan^{-1}(x_{i-1}) \\ \frac{dy_i}{dt} &= -\frac{1}{c}\left(x_i + by_i + a\right) \\ &\quad (i = 1,\ 2,\ \cdots,\ n,\ x_0 \equiv x_n) \end{aligned} \tag{1}$$

where $\delta(t)$ is the Dirac's delta function, h and τ are the amplitude and the period of the impulsive force, respectively. The system parameters are fixed as

$$a = 0.7,\ b = 0.8,\ c = 3.0 \tag{2}$$

for the occurrence of a stable equilibrium point in the unforced system of Eq. (1).

Although the single neuron without the external force has only one equilibrium point, combining these n ($n \geq 3$) neurons unidirectionally in a ring, n-phase periodic solutions are generated. Applying the impulsive forces, successive period-doubling bifurcations of the n-phase solution occur and chaotic states, namely n-phase chaos, appear. When $n = 2$ and 3, we find the parameter regions in which the switching phenomena of burst firing are observed. Moreover, the mechanisms of the switching phenomena are clarified by numerical bifurcation analysis.

2 Method of Analysis

In this section we summarize the analysis method for a bifurcation problem in a discontinuous system [11].

We describe the solution of Eq. (1) as

$$\begin{aligned} x(t) &= \varphi(t,\ x^0,\ y^0,\ \lambda) \\ y(t) &= \phi(t,\ x^0,\ y^0,\ \lambda) \end{aligned} \tag{3}$$

where $x, y \in R^n$ and $\lambda \in R^m$ denote n-dimensional states and an m-dimensional system parameter, respectively; φ and ϕ satisfy initial values:

$$\begin{aligned} x(0) &= \varphi(0,\ x^0,\ y^0,\ \lambda) = x^0 \\ y(0) &= \phi(0,\ x^0,\ y^0,\ \lambda) = y^0. \end{aligned} \tag{4}$$

Let us construct a Poincaré mapping correlated with Eq. (1).

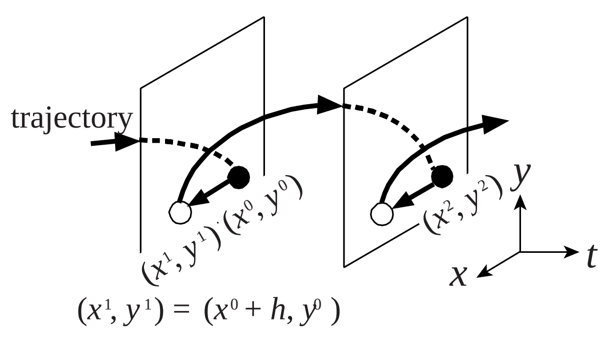


Figure 1: Schematic diagram of a trajectory observed in Eq. (1).

Firstly, we define a mapping T_1 which the translates the x-coordinate as

$$\begin{aligned} T_1 : R^{2n} &\rightarrow R^{2n} \\ (x^0, y^0) &\mapsto (x^1, y^1) = T_1(x^0, y^0) = (x^0 - h,\ y^0). \end{aligned} \tag{5}$$

Next we define an ordinary stroboscopic mapping T_2:

$$\begin{aligned} T_2 : R^{2n} &\rightarrow R^{2n} \\ (x^1,\ y^1) &\mapsto (x^2,\ y^2) = T_2(x^1,\ y^1) \end{aligned} \tag{6}$$

where

$$\begin{aligned} x^2 &= \varphi(\tau,\ x^1,\ y^1,\ \lambda) \\ y^2 &= \phi(\tau,\ x^1,\ y^1,\ \lambda). \end{aligned} \tag{7}$$

Then a Poincaré mapping T can be defined as the composition of T_1 and T_2:

$$\begin{aligned} T : R^{2n} &\rightarrow R^{2n} \\ (x^0, y^0) &\mapsto (x^2, y^2) = T(x^0, y^0) = T_2 \circ T_1(x^0, y^0). \end{aligned} \tag{8}$$

Note that T_1 and T_2 are differentiable so that T is also differentiable. A schematic diagram of a trajectory observed in Eq. (1) is shown in Fig. 1.

We assume that Eq. (1) has a fixed point of T. Let the fixed point be $u = (x, y) \in R^{2n}$. The fixed point satisfies

$$\begin{aligned} x &= \varphi(\tau,\ x - h,\ y,\ \lambda) \\ y &= \phi(\tau,\ x - h,\ y,\ \lambda). \end{aligned} \tag{9}$$

This fixed point of T corresponds to a periodic solution of Eq. (1) with period τ (period of the impulsive force), and similarly a p-periodic point of T corresponds to a periodic solution with period $p\tau$.

The characteristic equation of the fixed point u is described by

$$\det(\mu I - DT(u)) = 0 \tag{10}$$

where I is identity matrix and $DT(u)$ denotes the derivative of T evaluated at u. The bifurcations of the fixed point are given under the conditions:

(i) tangent bifurcation ($\mu = 1$)

(ii) period-doubling bifurcation ($\mu = -1$)

(iii) Neimark-Sacker bifurcation ($\mu = e^{j\theta}$)

(iv) D-type of branching (or pitchfork bifurcation) ($\mu = 1$ and symmetrical property).

We can obtain bifurcation parameter sets of the fixed point solving Eqs. (9) and (10) simultaneously [12].

In a bifurcation diagram shown in Sec. 3, tangent, period-doubling, Neimark-Sacker bifurcation and D-type of branching (pitchfork bifurcation) sets of m-periodic point are indicated, respectively, by symbols ${}_kG^m$, ${}_kI^m$, ${}_kN^m$ and ${}_kD^m$ where k denotes the number to distinguish several bifurcation sets of the same period.

For oscillatory solutions we use symbols L and S representing the continuation of pulses assigned by the symbol L and non-pulses assigned by the symbol S. More precisely we consider a wave of response as a pulse if the maximum value of wave x_i is more than one [9].

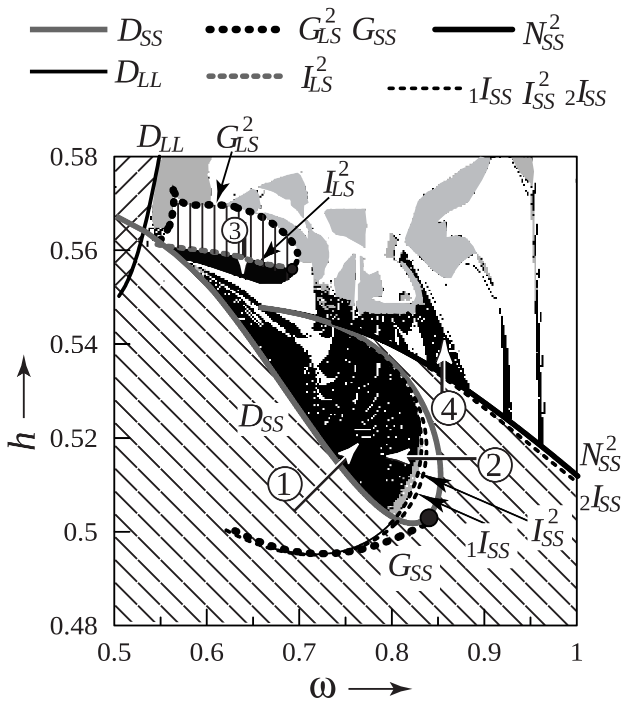


Figure 2: Bifurcation diagram of Eq. (1) with $n = 2$ and $W_1 = W_2 = -0.4$. Points marked by closed circle indicate codimension-two bifurcation points. Chaotic responses are observed in the region shaded by and . White regions correspond to periodic solutions.

3 Results

We consider bifurcation problems of Eq. (1) in parameter plane (ω, h), where $\omega = 2\pi/\tau$.

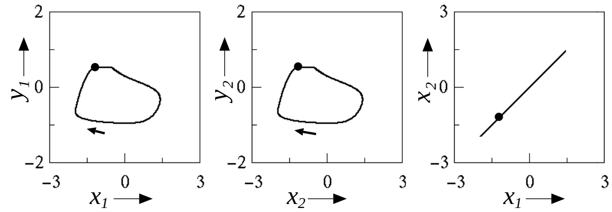
3.1 The case of $n = 2$

Figure 2 shows a bifurcation diagram of Eq. (1) with $n = 2$. In the shaded region , and we observe a stable LL (Fig. 3(a)), SS periodic solution (Fig. 3(b)) and a stable LS 2-periodic solution (Fig. 3(c)), respectively. D-type of branching sets D_{LL} and D_{SS} of the LL and the SS periodic solution, respectively, appear as a result of coupling two identical neurons which causes the symmetrical property. (Note that when $n = 2$, Eq. (1) is invariant under the interchange of the state variables.)

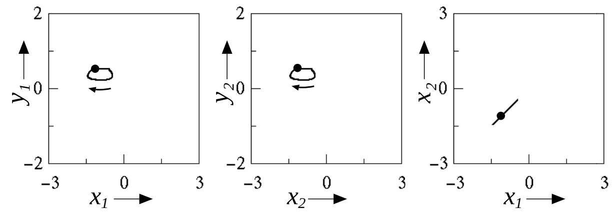
In the region shaded by , the switching chaos as shown in Fig. 4 is observed. Considering the pulses of Fig. 4 it seems that the burst firing is almost synchronized in the opposite phase. We study the mechanism of generation of the switching chaos by numerical bifurcation analysis. In Fig. 2, there exist several routes to the switching chaos: e.g. (the circled numbers correspond to those of Fig. 2)

① **intermittency route:** the SS periodic solution becomes unstable as a result of the pitchfork bifurcation D_{SS} and the switching chaos appears,

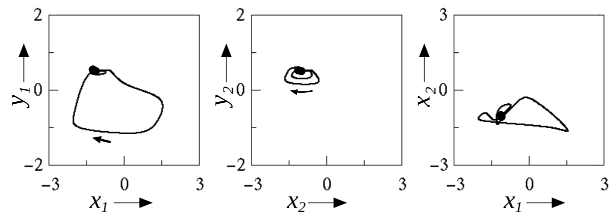
② **successive period-doubling bifurcations I:** the asymmetrical SS solution generated by D_{SS} meets successive period-doubling bifurcations (in Fig. 2 we



(a) LL periodic solution.



(b) SS periodic solution.



(c) LS 2-periodic solution.

Figure 3: Periodic solutions observed in the shaded regions (a), (b) and (c) of Fig. 2. Arrows and the points marked by closed circles indicate the time direction of the trajectory and the fixed point of the Poincaré map T, respectively. (Left) Neuron 1. (Middle) Neuron 2. (Right) Neuron 1 vs. Neuron 2.

only show two period-doubling bifurcations ${}_1I_{SS}$ and I^2_{SS}) and the generated two chaotic attractors merge into one,

③ **successive period-doubling bifurcations II:** the SL 2-periodic solution pass through the successive period-doubling bifurcations (in Fig. 2 the first period-doubling bifurcation I^2_{LS} is shown) and the switching chaos is generated,

④ **collapse of quasi-periodic solution:** the 2-periodic SS solution generated by ${}_2I_{SS}$ meets Neimark-Sacker bifurcation N^2_{SS} and the generated quasi-periodic solution collapses.

This burst firing can be observed stably in a wide parameter range and also observed in the system broken the symmetrical property.

3.2 The case of $n = 3$

The single BVP neuron without the external force has only one stable equilibrium point. We know that coupling these identical n ($n \geq 3$) BVP neurons unidirectionally, a stable n-phase periodic oscillation is observed [13].

Therefore when $n = 3$ in Eq. (1), there exists a three-phase solution. The stable three-phase solution meets successive period-doubling bifurcations and almost three-phase chaos as shown in Fig. 5 is generated.

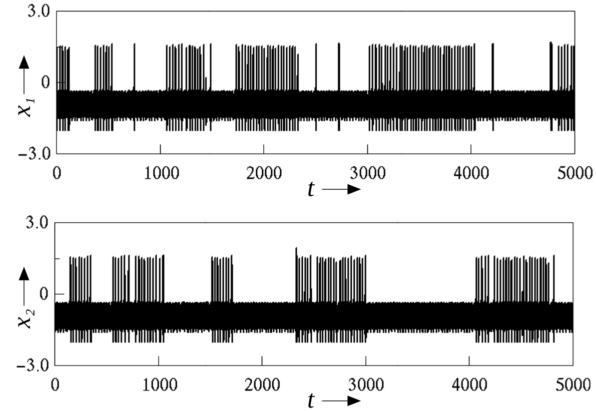


Figure 4: Time series of burst firing. $n = 2$. $\omega = 0.63$. $h = 0.55$. $W_1 = -0.4$. $W_2 = -0.4$.

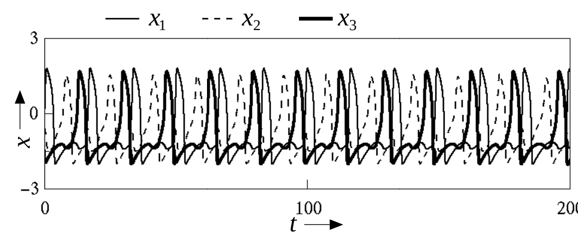


Figure 5: Time series of chaos generated by successive period-doubling bifurcations of a three-phase solution. $n = 3$. $\omega = 0.38$. $h = 0.616$. $W_1 = -0.4$. $W_2 = -0.4$. $W_3 = -0.4$.

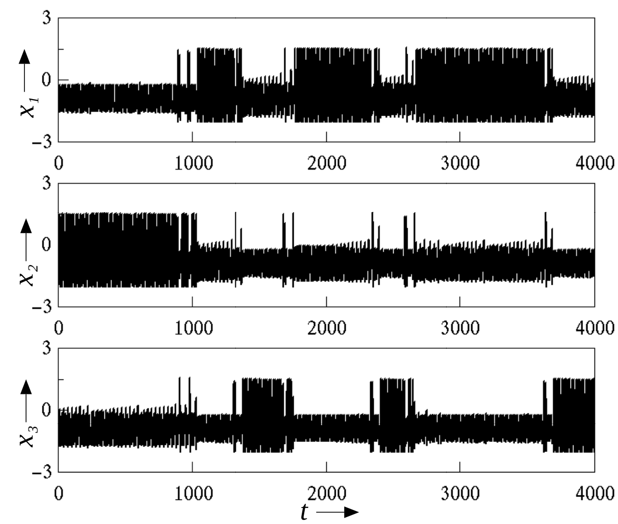


Figure 6: Time series of burst firing. $n = 3$. $\omega = 0.423$. $h = 0.5626$. $W_1 = -0.4$. $W_2 = -0.4$. $W_3 = -0.4$.

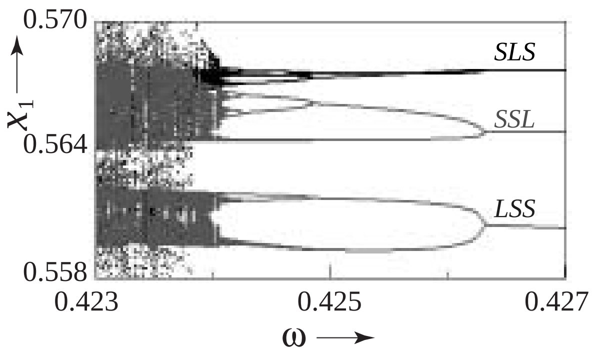


Figure 7: Bifurcation diagram of LSS, SLS and SSL periodic solutions.

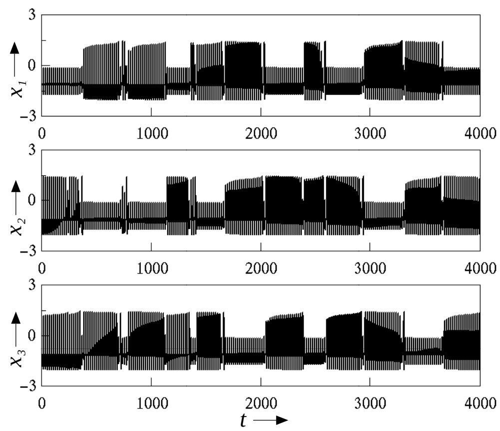


Figure 8: Time series of burst firing. $n = 3$. $\omega = 0.34$. $h = 0.5648$. $W_1 = -0.4$. $W_2 = -0.4$. $W_3 = -0.4$.

Figure 6 shows time series of LSS-type switching chaos, where LSS means that i, $(i+1)$ and $(i+2)$-th ($i = 1, 2, 3$, $(i{+}1)$ and $(i+2)$ mod 3) neurons generate one, zero and zero pulse in one period of the external force, respectively. For example, between $t = 0$ and $t \simeq 800$ the neuron 2 (x_2) generates one pulse in one period of the external force, while the neuron 1 (x_1) and the neuron 3 (x_3) do not generate any pulses. The bifurcation mechanism of generating the switching chaos is shown in Fig. 7. The LSS-periodic solution meets successive period-doubling bifurcations (by the symmetrical property of Eq. (1), the period-doubling bifurcations of SLS and SSL-periodic solutions occur at the same parameter values) and the LSS, SLS and SSL chaotic solutions appear. After the crisis of these three types of chaotic solutions, switching chaos is generated. In Fig. 8 we show LLS-type switching chaos generated by the bifurcation mechanism similar to that of LSS-type switching chaos.

4 Concluding Remarks

We have investigated bifurcation and chaos observed in coupled BVP neurons with external impulsive forces. Although the single neuron without the external force has only one equilibrium point, combining these n neurons unidirectionally in a ring, n-phase periodic solutions are generated. Applying the impulsive forces, successive period-doubling bifurcations of the n-phase solution occur and chaotic states, namely n-phase chaos, appear. When $n = 2$ and 3, we found the parameter regions in which the switching phenomena of burst firing are observed. Moreover, the mechanisms of the switching phenomena are clarified by numerical bifurcation analysis.

These switching phenomena were observed in the lobster stomatogastric ganglion [14]. Therefore it is interesting open problems to study the mechanisms of generating such phenomena in asymmetrical neural networks.

Acknowledgments

H. Kitajima would like to thank Dr. T. Yoshinaga of the University of Tokushima and Dr. T. Nomura of Osaka University for their helpful comments.

References

[1] K. Aihara and G. Matsumoto, "Chaotic oscillations and bifurcations in squid giant axons," in *Chaos*, ed. by A.V. Holden, Manchester University Press, Manchester and Princeton University Press, Princeton, pp.257–269, 1986.

[2] J.P. Segundo, M. Stiber and J.-F. Vibert, "Synaptic coding of spike trains," in *The Handbook of Brain Theory and Neural Networks*, ed. M.A. Arbib, The MIT Press, Cambridge, pp.953–956, 1995

[3] S.K. Han, S.H. Park, T.G. Yim, S. Kim and S. Kim, "Chaotic bursting behavior of coupled neural oscillators," *Int. J. Bifurcation and Chaos*, vol.7, no.4, pp.877–888. 1997

[4] S.G. Lee, S. Kim, and H. Kook, "Synchrony and clustering in two and three synaptically coupled Hodgkin-Huxley neurons with time delay," *Int. J. Bifurcation and Chaos*, vol.7, no.4, pp.889–896. 1997

[5] H. Kitajima, T. Yoshinaga, K. Aihara and H. Kawakami, "Chaotic bursts and bifurcation in chaotic neural networks with ring structure," *Int. J. Bifurcation and Chaos*, (accepted).

[6] B. Barnes and R. Grimshaw, "Numerical studies of the periodically forced Bonhoeffer van der Pol system," *Int. J. Bifurcation and Chaos*, vol.7, no.12, pp.2653–2689, Dec. 1997.

[7] S. Doi and S. Sato, "The global bifurcation structure of the BVP neuronal model driven by periodic pulse trains," *Mathematical Biosciences*, vol.125, pp.229–250, 1995.

[8] T. Nomura, S. Sato, S. Doi, J.P. Segundo and M.D. Stiber, "Global Bifurcation Structure of a Bonhoeffer-van der Pol Oscillator Driven by Periodic Pulse Trains," *Biological Cybernetics*, vol.72, pp.55–67, 1994.

[9] T. Yoshinaga and H. Kawakami, "Bifurcation in a BVP Equation with Periodic Impulsive Force," *Proc. NOLTA'95*, pp.331–334, 1995.

[10] K. Tsumoto, T. Yoshinaga and H. Kawakami, "Bifurcation of periodic solutions observed in Two BVP Neurons coupled by synaptic transmission," *Technical Report of IEICE*, NLP99-43, pp.115–122, June 1999.

[11] O. Morimoto and H. Kawakami, "Bifurcation diagram of a BVP equation with impulsive external force," *Proc. NOLTA'94*, pp.205–208, 1994.

[12] H. Kawakami, "Bifurcation of periodic responses in forced dynamic nonlinear circuits: computation of bifurcation values of the system parameters," *IEEE Trans. Circuits & Syst.* vol.CAS-31, no.3, pp.248–260 March 1984.

[13] H. Kitajima and H. Kawakami, "Bifurcation of a unidirectionally coupled oscillator," *Technical Report of IEICE*, NLP96-144, pp.25–32, Feb. 1997.

[14] J.P. Miller and A.I. Selverston, "Neural mechanisms for the production of the lobster pyloric motor pattern," in *Model Neural Networks and Behavior*, ed. by A.I. Selverston, Plenum Press, New York, 1985.