Bifurcation of Periodic States in Coupled BVP Neurons with External Impulsive Forces

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Abstract— We investigate bifurcation and chaos observed in coupled BVP neurons with external impulsive forces. Although the single neuron without the external force has only one equilibrium point, combining these n ($n \geq 3$) neurons unidirectionally in a ring, n-phase periodic solutions are generated. Applying the impulsive forces, successive period-doubling bifurcations of the n-phase solution occur and chaotic states, namely n-phase chaos, appear. When $n = 2$ and 3, we find the parameter regions in which the switching phenomena of burst firing are observed. Moreover, the mechanisms of the switching phenomena are clarified by numerical bifurcation analysis.

I. Introduction

Many investigators have studied various temporal patterns of spikes observed in the brain [1–3]. Among such firing patterns, synchronous firing of neurons in connection with neuronal signal processing has attracted much interest [4, 5]. It is important to consider global dynamics of networks composed of nonlinear neurons in order to clarify not only mechanisms of synchronous firing of neurons but also its functional roles.

In this paper we examine nonlinear dynamics and bifurcations of unidirectionally coupled Bonhöffer van der Pol (BVP) neurons with external impulsive forces. The BVP equation is a well-known neuron model representing the electrical behavior across a nerve membrane and has been widely studied [6–10].

The model equation analyzed in this paper is described as

$$\begin{aligned}\frac{dx_i}{dt} &= c\left(x_i - \frac{1}{3}x_i^3 + y_i\right) - h\sum_{k\in Z}\delta(t-k\tau) \\ &\quad + W_i \tan^{-1}(x_{i-1}) \\ \frac{dy_i}{dt} &= -\frac{1}{c}\left(x_i + by_i - a\right) \\ &\quad (i = 1,\ 2,\ \cdots,\ n,\ x_0 \equiv x_n)\end{aligned} \tag{1}$$

where $\delta(t)$ is the Dirac's delta function, h and τ are the amplitude and the period of the impulsive force, respectively. The system parameters are fixed as

$$a = 0.7,\ b = 0.8,\ c = 3.0 \tag{2}$$

for the occurrence of a stable equilibrium point in the unforced system of Eq. (1).

Although the single neuron without the external force has only one equilibrium point, combining these n neurons unidirectionally in a ring, n-phase periodic solutions are generated. Applying the impulsive forces, successive period-doubling bifurcations of the n-phase solution occur and chaotic states, namely n-phase chaos, appear. When $n = 2$ and 3, we find the parameter regions in which the switching phenomena of burst firing are observed. Moreover, the mechanisms of the switching phenomena are clarified by numerical bifurcation analysis.

II. Method of Analysis

In this section we summarize the analysis method for a bifurcation problem in a discontinuous system [11].

We describe the solution of Eq. (1) as

$$\begin{aligned}\boldsymbol{x}(t) &= \boldsymbol{\varphi}(t,\ \boldsymbol{x}^0,\ \boldsymbol{y}^0,\ \boldsymbol{\lambda}) \\ \boldsymbol{y}(t) &= \boldsymbol{\phi}(t,\ \boldsymbol{x}^0,\ \boldsymbol{y}^0,\ \boldsymbol{\lambda})\end{aligned} \tag{3}$$

where $\boldsymbol{x}, \boldsymbol{y} \in R^n$ and $\boldsymbol{\lambda} \in R^m$ denote n-dimensional states and an m-dimensional system parameter, respectively; $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$ satisfy initial values:

$$\begin{aligned}\boldsymbol{x}(0) &= \boldsymbol{\varphi}(0,\ \boldsymbol{x}^0,\ \boldsymbol{y}^0,\ \boldsymbol{\lambda}) = \boldsymbol{x}^0 \\ \boldsymbol{y}(0) &= \boldsymbol{\phi}(0,\ \boldsymbol{x}^0,\ \boldsymbol{y}^0,\ \boldsymbol{\lambda}) = \boldsymbol{y}^0.\end{aligned} \tag{4}$$

Let us construct a Poincaré mapping correlated with Eq. (1). Firstly, we define a mapping T_1 which translates x-coordinate as

$$\begin{aligned}&T_1 : R^{2n} \to R^{2n} \\ &\quad (\boldsymbol{x}^0,\ \boldsymbol{y}^0) \mapsto (\boldsymbol{x}^1, \boldsymbol{y}^1) = T_1(\boldsymbol{x}^0, \boldsymbol{y}^0) = (\boldsymbol{x}^0 - \boldsymbol{h},\ \boldsymbol{y}^0).\end{aligned} \tag{5}$$

Next we define an ordinary stroboscopic mapping T_2:

$$\begin{aligned}&T_2 : R^{2n} \to R^{2n} \\ &\quad (\boldsymbol{x}^1,\ \boldsymbol{y}^1) \mapsto (\boldsymbol{x}^2,\ \boldsymbol{y}^2) = T_2(\boldsymbol{x}^1,\ \boldsymbol{y}^1)\end{aligned} \tag{6}$$

where

$$\begin{aligned}\boldsymbol{x}^2 &= \boldsymbol{\varphi}(\tau,\ \boldsymbol{x}^1,\ \boldsymbol{y}^1,\ \boldsymbol{\lambda})\\ \boldsymbol{y}^2 &= \boldsymbol{\phi}(\tau,\ \boldsymbol{x}^1,\ \boldsymbol{y}^1,\ \boldsymbol{\lambda}).\end{aligned} \tag{7}$$

Then a Poincaré mapping T can be defined as the composition of T_1 and T_2:

$$\begin{aligned}&T : R^{2n} \to R^{2n} \\ &(\boldsymbol{x}^0,\ \boldsymbol{y}^0) \mapsto (\boldsymbol{x}^2, \boldsymbol{y}^2) = T(\boldsymbol{x}^0, \boldsymbol{y}^0) = T_2 \circ T_1(\boldsymbol{x}^0, \boldsymbol{y}^0).\end{aligned} \tag{8}$$

Note that T_1 and T_2 are differentiable so that T is also differentiable.

Let a fixed point of T be $\boldsymbol{u} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}$. The fixed point satisfies

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{\varphi}(\tau,\ \boldsymbol{x} - \boldsymbol{h},\ \boldsymbol{y},\ \boldsymbol{\lambda})\\ \boldsymbol{y} &= \boldsymbol{\phi}(\tau,\ \boldsymbol{x} - \boldsymbol{h},\ \boldsymbol{y},\ \boldsymbol{\lambda}).\end{aligned} \tag{9}$$

This fixed point of T corresponds to a periodic solution of Eq. (1) with period τ (period of the impulsive force), and similarly a p-periodic point of T corresponds to a periodic solution with period $p\tau$.

The characteristic equation of the fixed point $\boldsymbol{u}$ is described by

$$\det(\mu \boldsymbol{I} - \boldsymbol{DT}(\boldsymbol{u})) = 0 \tag{10}$$

where $\boldsymbol{I}$ is identity matrix and $\boldsymbol{DT}(\boldsymbol{u})$ denotes the derivative of T evaluated at $\boldsymbol{u}$. The bifurcations of the fixed point are given under the conditions:

(i) tangent bifurcation ($\mu = 1$)

(ii) period-doubling bifurcation ($\mu = -1$)

(iii) Neimark-Sacker bifurcation ($\mu = e^{j\theta}$)

(iv) D-type of branching (or pitchfork bifurcation) ($\mu = 1$ and symmetrical property).

We can obtain bifurcation parameter sets of the fixed point solving Eqs. (9) and (10) simultaneously [12].

In bifurcation diagrams tangent, period-doubling and D-type of branching (pitchfork bifurcation) sets are indicated by symbols G, I and D, respectively.

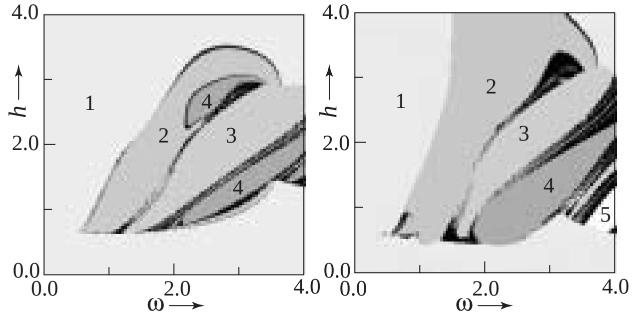
For oscillatory solutions we use symbols L and S representing the continuation of pulses assigned by the symbol L and non-pulses assigned by the symbol S. More precisely we consider a wave of response as a pulse if the minimum value of wave x_i is less than minus one [9].

III. Results

We consider bifurcation problems of Eq. (1) in parameter plane (ω, h), where $\omega = 2\pi/\tau$.

A. The case of $n = 1$

We show a bifurcation diagram of the single BVP neuron with the external impulsive force in Fig. 1. In this system we can observe the existence of a mean firing rate as a monotone increasing function of a system parameter and various kinds of solutions: chaotic oscillation caused by successive period-doubling bifurcations, quasi-periodic solutions generated by Neimark-Sacker bifurcations and periodic solutions of various orders of period [9].



(a) $n = 1$. (b) $n = 2$. $W_1 = W_2 = -0.4$.

Figure 1: Bifurcation diagram. In each region denoted by p, we can observe p-periodic solutions. The dynamics is chaotic (or quasi-periodic) in the region colored by black.

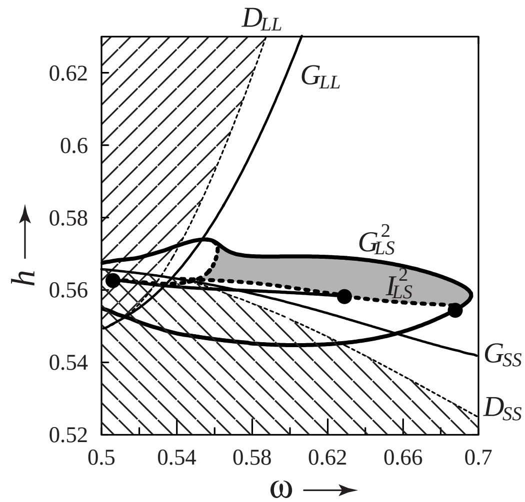
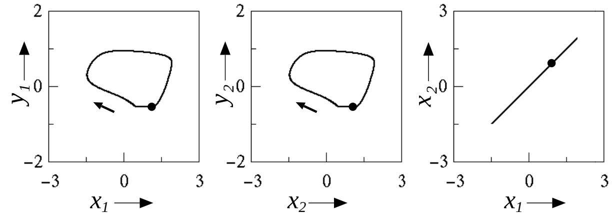


Figure 2: Enlarged diagram of a part of Fig. 1(b). Points marked by closed circle indicate codimension-two bifurcation points.

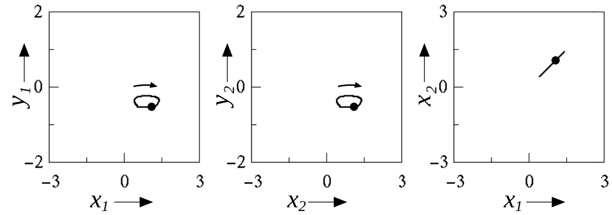
B. The case of $n = 2$

Figure 1(b) shows a bifurcation diagram of two-coupled BVP neurons. We can see that global bifurcation structure is similar to that of Fig. 1; however, in Fig. 1 (b) chaotic (or quasi-periodic) solutions are observed in a wider parameter range.

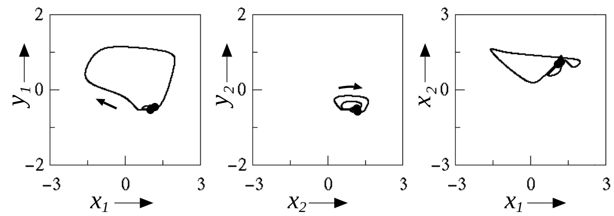
Figure 2 shows an enlarged diagram of a part of Fig. 1(b). In the shaded region and we observe a stable LL (Fig. 3(a)) and SS (Fig. 3(b)) periodic solution, respectively. D-type of branching sets D_{LL} and D_{SS} of the LL and the SS periodic solution, respectively, appear as a result of coupling two identical neurons which causes the symmetrical property.



(a) LL periodic solution.



(b) SS periodic solution.



(c) LS 2-periodic solution.

Figure 3: Periodic solutions observed in the shaded regions of Fig. 2. Arrows and the points marked by closed circles indicate the time direction of the trajectory and the fixed point of Poincaré map, respectively. (Left) Neuron 1. (Middle) Neuron 2. (Right) Neuron 1 vs. Neuron 2.

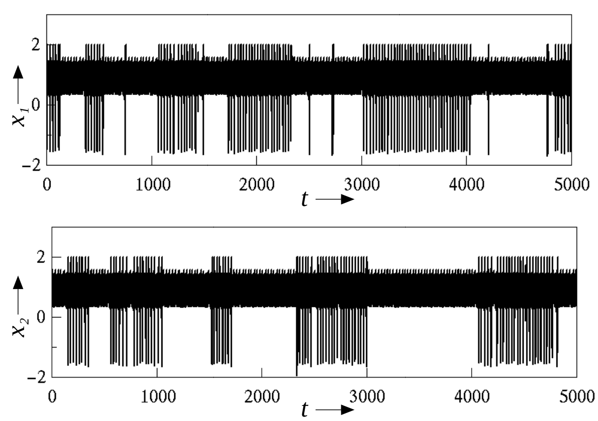


Figure 4: Time series of burst firing synchronized in the opposite phase. $n = 2$. $\omega = 0.63$. $h = 0.55$. $W_1 = -0.4$. $W_2 = -0.4$.

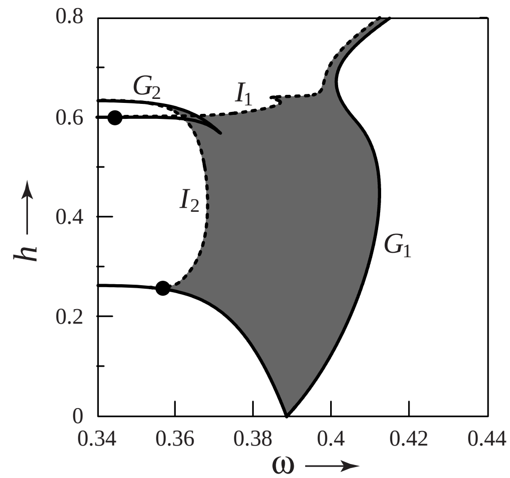


Figure 5: Bifurcation diagram of three-phase solutions. $n = 3$.

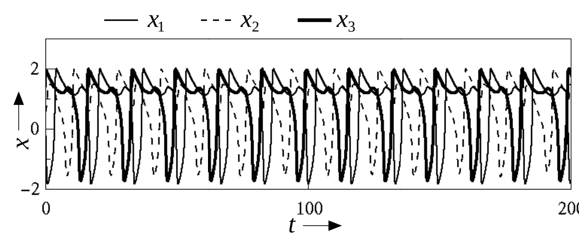


Figure 6: Time series of chaos generated by successive period-doubling bifurcations of a three-phase solution. $n = 3$. $\omega = 0.38$. $h = 0.616$. $W_1 = -0.4$. $W_2 = -0.4$. $W_3 = -0.4$.

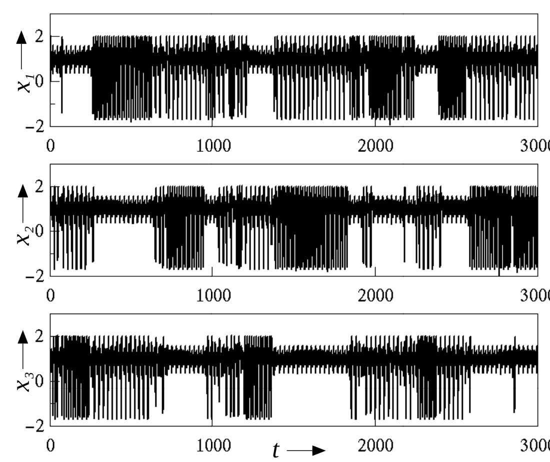


Figure 7: Time series of burst firing. $\omega = 0.688$. $h = 0.56$. $W_1 = -0.4$. $W_2 = -0.4$. $W_3 = -0.4$. $n = 3$.

(Note that when $n = 2$, Eq. (1) is invariant under the interchange of the state variables.) We observe a stable LS 2-periodic solution (Fig. 3(c)) in the shaded region . By the symmetrical property of the system a stable SL 2-periodic solution also exists in the same parameter region.

Decreasing the parameter value of h from this area, the LS 2-periodic solution meets the period-doubling bifurcation I^2_{LS} and becomes a chaotic solution by successive period-doubling bifurcations. By further decreasing of the value of h chaos as shown in Fig. 4 is obtained. Considering the pulses of Fig. 4 it seems that the burst firing is almost synchronized in the opposite phase. This burst firing can be observed stably in a wide parameter range and also observed in the system broken the symmetrical property.

C. The case of $n = 3$

The single BVP neuron without the external force has only one stable equilibrium point. We know that coupling these identical n ($n \geq 3$) BVP neurons unidirectionally, a stable n-phase periodic oscillation is observed [13].

Therefore when $n = 3$ in Eq. (1), there exists a three-phase solution. Figure 5 shows a bifurcation diagram of three-phase solutions. In Fig. 5 the tangent bifurcation set G_1 of the three-phase solution meets the axis of $h = 0$ at $\omega \simeq 0.388$ which corresponds to the natural frequency of the three-phase solution in the unforced system ($h = 0$ in Eq. (1)). The stable three-phase solution exists in the region shading by . By increasing the value of h successive period-doubling bifurcations (only the first bifurcation I_1 is shown in Fig. 5) occur and almost three-phase chaos as shown in Fig. 6 is generated.

Figure 7 shows time series of SLL^2 chaotic solution, where SLL^2 means that i, $(i+1)$ and $(i+2)$-th (i mod 3) neurons generate zero, one and two pulses in one period of the external force, respectively. For example, between $t \simeq 250$ and $t \simeq 600$ the neuron 3 (x_3) and 1 (x_1) generate one and two pulses, respectively, in one period of the external force, while the neuron 2 (x_2) does not generate any pulses.

IV. Concluding Remarks

We have investigated bifurcation and chaos observed in coupled BVP neurons with external impulsive forces. Although the single neuron without the external force has only one equilibrium point, combining these n neurons unidirectionally in a ring, n-phase periodic solutions are generated. Applying the impulsive forces, successive period-doubling bifurcations of the n-phase solution occur and chaotic states, namely n-phase chaos, appear. When $n = 2$ and 3, we find the parameter regions in which the switching phenomena of burst firing are observed. Moreover, the mechanisms of the switching phenomena are clarified by numerical bifurcation analysis.

To analyze the case of a large number of coupled neurons and to classify observed chaos using coefficient of variation of interspike intervals (ISI) [14] are interesting problems open to the future.

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References

[1] K. Aihara and G. Matsumoto, "Chaotic oscillations and bifurcations in squid giant axons," in *Chaos*, ed. by A.V. Holden, Manchester University Press, Manchester and Princeton University Press, Princeton, pp.257–269, 1986.

[2] J.P. Segundo, M. Stiber and J.-F. Vibert, "Synaptic coding of spike trains," in *The Handbook of Brain Theory and Neural Networks*, ed. M.A. Arbib, The MIT Press, Cambridge, pp.953–956, 1995

[3] J. Bargas and E. Galarraga, "Ion channels : keys to neuronal specialization," in *The Handbook of Brain Theory and Neural Networks*, ed. M.A. Arbib, The MIT Press, Cambridge, pp.496–501, 1995

[4] S.K. Han, S.H. Park, T.G. Yim, S. Kim and S. Kim, "Chaotic bursting behavior of coupled neural oscillators," *Int. J. Bifurcation and Chaos*, vol.7, no.4, pp.877–888. 1997

[5] S.G. Lee, S. Kim, and H. Kook, "Synchrony and clustering in two and three synaptically coupled Hodgkin-Huxley neurons with time delay," *Int. J. Bifurcation and Chaos*, vol.7, no.4, pp.889–896. 1997

[6] B. Barnes and R. Grimshaw, "Numerical studies of the periodically forced Bonhoeffer van der Pol system," *Int. J. Bifurcation and Chaos*, vol.7, no.12, pp.2653–2689, Dec. 1997.

[7] S. Doi and S. Sato, "The global bifurcation structure of the BVP neuronal model driven by periodic pulse trains," *Mathematical Biosciences*, vol.125, pp.229–250, 1995.

[8] T. Nomura, S. Sato, S. Doi, J.P. Segundo and M.D. Stiber, "Global Bifurcation Structure of a Bonhoeffer-van der Pol Oscillator Driven by Periodic Pulse Trains," *Biological Cybernetics*, vol.72, pp.55–67, 1994.

[9] T. Yoshinaga and H. Kawakami, "Bifurcation in a BVP Equation with Periodic Impulsive Force," *Proc. NOLTA'95*, pp.331–334, 1995.

[10] K. Tsumoto, T. Yoshinaga and H. Kawakami, "Bifurcation of periodic solutions observed in Two BVP Neurons coupled by synaptic transmission," *Technical Report of IEICE*, NLP99-43, pp.115–122, June 1999.

[11] O. Morimoto and H. Kawakami, "Bifurcation diagram of a BVP equation with impulsive external force," *Proc. NOLTA'94*, pp.205–208, 1994.

[12] H. Kawakami, "Bifurcation of periodic responses in forced dynamic nonlinear circuits: computation of bifurcation values of the system parameters," *IEEE Trans. Circuits & Syst.* vol.CAS-31, no.3, pp.248–260 March 1984.

[13] H. Kitajima and H. Kawakami, "Bifurcation of a unidirectionally coupled oscillator," *Technical Report of IEICE*, NLP96-144, pp.25–32, Feb. 1997.

[14] W.R. Softky and C. Koch, "The Highly Irregular Firing of Cortical Cells Is Inconsistent with Temporal Integration of Random EPSPs," *Neuroscience*, 13(1), pp.334–350, Jan. 1993.