International Symposium on Nonlinear Theory and its Applications
Xi'an, PRC, October 7–11, 2002

Synchronized states observed in coupled four oscillators

Hiroyuki KITAJIMA[†] and Hiroshi KAWAKAMI[‡]

[†]Kagawa University
2217-20 Hayashi, Takamatsu, Kagawa, Japan
Phone:+81-87-864-2226, Fax:+81-87-864-2262
Email: kitaji@eng.kagawa-u.ac.jp

[‡]Tokushima University
2-1 Minami-Josanjima, Tokushima, Japan
Phone:+81-88-656-7465, Fax:+81-88-655-0747
Email: kawakami@ee.tokushima-u.ac.jp

1. Introduction

Systems of coupled oscillators are widely used as models for biological rhythmic oscillations such as human circadian rhythms[1, 2], finger movements, animal locomotion[3], swarms of fireflies that flash in synchrony, synchronous firing of cardiac pacemaker cells[5, 6], and so on. Using these coupled oscillator models, many investigators have studied the mechanism of generation of synchronous oscillation and phase transitions between distinct oscillatory modes. From the standpoint of bifurcation, the former and the latter correspond to Hopf bifurcation of an equilibrium point (or tangent bifurcation of a periodic solution) and pitchfork bifurcation (or period-doubling bifurcation) of a periodic solution, respectively. Using group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning with the bifurcations of equilibrium points and periodic solutions[7].

In the study of coupled oscillator system, the four-coupled oscillator system is one of the most interesting system, because there exists an irregular degenerate oscillatory mode (or an independent pair of anti-phase mode) [8, 9] when the equation of the single oscillator is invariant under inversion of state variables.

Mishima and Kawakami studied the oscillatory modes generated by the Hopf bifurcations of the origin (equilibrium point) in several systems of coupled four BVP (Bonhöffer-van der Pol) oscillators [10]. However, they considered the Hopf bifurcation of the origin, because only the Hopf bifurcation of the origin is supercritical. Tsumoto *et al*. investigated bifurcations of the Modified BVP (MBVP) equation[11]. In the MVBP system, the supercritical Hopf bifurcation of non-origin equilibrium points occurs.

In this paper, we examine the oscillatory modes generated by the Hopf bifurcations of non-origin equilibrium points in the four-coupled oscillator system . The Hopf bifurcations of the equilibrium points with strong symmetrical property and the generated oscillatory modes are classified. We observe three-phase, four-phase, in-phase and a pair of anti-phase synchronized states. The three-phase and four-phase solutions meet the Neimark-Sacker bifurcations, and three-phase and four-phase quasi-periodic solutions are generated, respectively.

2. Circuit Equation

We consider the coupled MBVP oscillator system shown in Fig. 1. The circuit equation is described as

$$\begin{aligned} L_1\frac{di_{k1}}{dt} &= -R_1 i_{k1} - v_k \\ L_2\frac{di_{k2}}{dt} &= -R_2 i_{k2} - v_k \\ C\frac{dv_k}{dt} &= i_{k1} + i_{k2} - g(v_k) \\ &\quad -G(3v_k - v_{k+1} - v_{k-1} - v_{k+2}) \\ &(k=1,\cdots,4,\ v_0\equiv v_4,\ v_5\equiv v_1,\ v_6\equiv v_2), \end{aligned} \tag{1}$$

where the nonlinear conductance $g(v_k)$ is assumed to be

$$g(v_k) = -v_k + \frac{1}{3}v_k^3. \tag{2}$$

The values of system parameters are fixed as [11]

$$L_1^{-1} = 0.2,\ R_1 = 4.0,\ R_2 = 2.1,\ C^{-1} = 3.0. \tag{3}$$

The Jacobi matrix of Eq. (1) is described by

$$DF = \begin{bmatrix} X_0 & X_1 & X_1 & X_1 \\ X_1 & X_0 & X_1 & X_1 \\ X_1 & X_1 & X_0 & X_1 \\ X_1 & X_1 & X_1 & X_0 \end{bmatrix}. \tag{4}$$

Each block is given by

$$X_0 = \begin{bmatrix} -R_1L_1^{-1} & 0 & -L_1^{-1} \\ 0 & -R_2L_2^{-1} & -L_2^{-1} \\ C^{-1} & C^{-1} & C^{-1}(1-v_{*k}^2) - 3d \end{bmatrix},$$

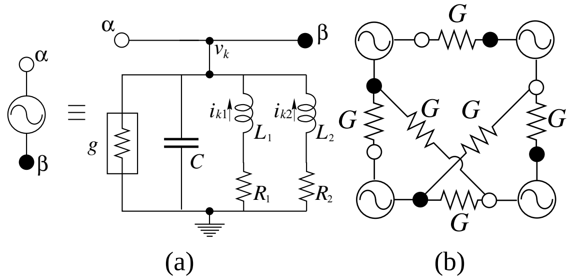


Figure 1: MBVP circuit (a) and coupled system (b).

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} \quad (5)$$

where v_{*k} is a coordinate of an equilibrium point and $d = C^{-1}G$. Using orthogonal matrix given by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2}\, I & I & O & 1/\sqrt{2}\, I \\ 1/\sqrt{2}\, I & O & I & -1/\sqrt{2}\, I \\ 1/\sqrt{2}\, I & -I & O & 1/\sqrt{2}\, I \\ 1/\sqrt{2}\, I & O & -I & -1/\sqrt{2}\, I \end{bmatrix} \quad (6)$$

we diagonalize the Jacobian matrix (4) as:

$$Q^{-1} \cdot DF \cdot Q = \begin{bmatrix} Y_0 & O & O & O \\ O & Y_1 & O & O \\ O & O & Y_1 & O \\ O & O & O & Y_1 \end{bmatrix} \quad (7)$$

where

$$Y_0 = X_0 + 3X_1, \quad (8)$$
$$Y_1 = X_0 - X_1, \quad (9)$$
$$(10)$$

In the next section we classify the oscillatory modes generated by the Hopf bifurcation in each block Y_l (l = 1, 2) in Eq. (7).

3. Results

Table 1: Classification of oscillatory modes.

equilibrium point	oscillatory modes
(a_1, a_1, a_1, a_1)	4-phase, in-phase(1)
$(a_2, a_2, -a_2, -a_2)$	a pair of anti-phase, a pair of in-phase
(a_3, a_3, a_3, b)	3-phase, in-phase(2)

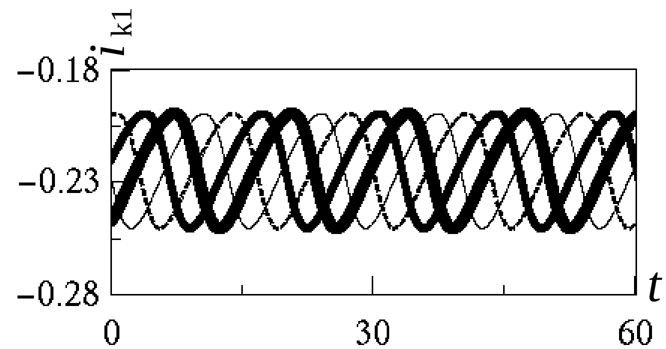


Figure 2: Waveforms of a 4-phase synchronized state. $L_2^{-1} = 0.035$. $d = -0.01$.

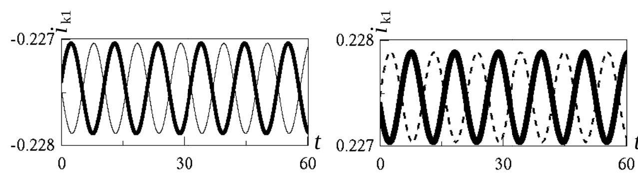


Figure 3: Waveforms of a pair of an anti-phase synchronized state. $L_2^{-1} = 0.048$. $d = -0.0016$.

In Eq. (1) there exists three equilibrium points satisfying $v_{*1}^2 = v_{*2}^2 = v_{*3}^2 = v_{*4}^2$; those are $(v_{*1}, v_{*2}, v_{*3}, v_{*4})$ = (0, 0, 0, 0), (a_1, a_1, a_1, a_1) and $(a_2, a_2, -a_2, -a_2)$. Here we study the supercritical Hopf bifurcations of equilibrium points (a_1, a_1, a_1, a_1) and (a_2, a_2, $-a_2$, $-a_2$). And also (a_3, a_3, a_3, b) type equilibrium point is studied, because this type has the supercritical Hopf bifurcations.

In Tab. 1 we classify oscillatory modes generated by the Hopf bifurcations for three types of equilibrium points. For each equilibrium point, an in-phase solution is generated by the Hopf bifurcation of the block Y_0 in Eq. (7). On the other hand, the Hopf bifurcations of the blocks Y_1 are degenerate (three pairs of eigenvalues satisfy the condition of the Hopf bifurcation). Thus many oscillatory modes (stable and unstable) appear simultaneously by the degenerate Hopf bifurcation.

Waveforms of four-phase, a pair of anti-phase, three-phase and in-phase(2) solutions are shown in Figs. 2, 3, 4 and 5, respectively. In each waveform, there is no correspondence between the kinds of curved line and the order of oscillators denoted by k, because the oscillators are fully connected and we can exchange any oscillators.

From Fig. 4 we can see that three oscillators synchronized at three-phase (Fig. 4(a)) and oscillation of one oscillator is almost stopped (Fig. 4(b)). On the other hand, in Fig. 5 three oscillators are synchronized at in-phase but their oscillations are almost stopped (Fig. 5(a)), and only one oscillator oscil-

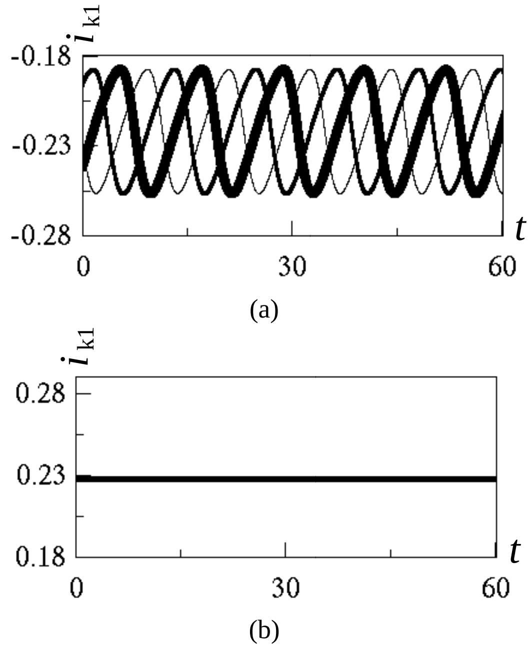


Figure 4: Waveforms of a 3-phase synchronized state. $L_2^{-1} = 0.06$. $d = -0.0016$.

lates (Fig. 5(b)).

By changing the values of parameter L_2^{-1} and d, the three-phase and four-phase solutions meet the Neimark-Sacker bifurcations, and three-phase and four-phase quasi-periodic solution appear as shown in Fig. 6 and 7, respectively.

Figures 3 and 9 show some interesting oscillatory modes generated by the degenerate Hopf bifurcations as mentioned earlier. In Fig. 3 the oscillations of two oscillators completely stopped and other two oscillators are synchronized at in-phase. In Fig. 9 three oscillators are synchronized at in-phase with small amplitude, and one oscillator is synchronized at anti-phase against other three ones and has large amplitude.

4. Conclusion

We have studied the oscillatory modes generated by the Hopf bifurcations in coupled four oscillators. The Hopf bifurcations of three types of equilibrium points and the generated oscillatory modes are classified. Moreover, by numerical bifurcation analysis we observed various interesting synchronized states caused by the degenerate Hopf bifurcations.

Considering the associative memory model for storing patterns as oscillatory states[12], this system has the advantage of many oscillatory modes.

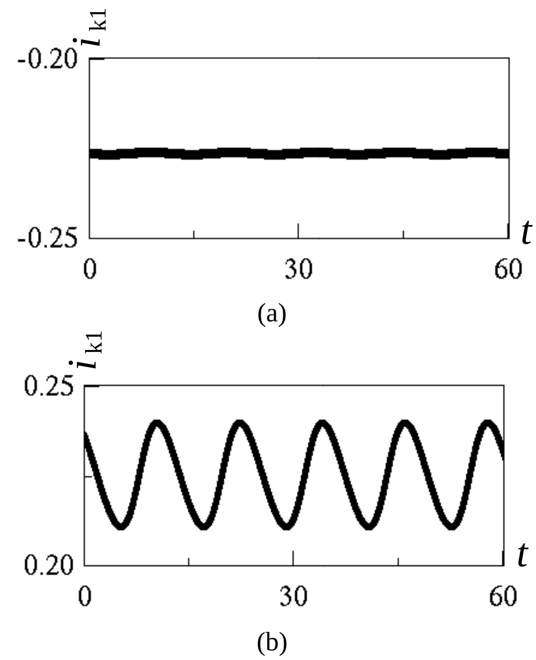


Figure 5: Waveforms of an in-phase synchronized state caused by the Hopf bifurcation of (a_3, a_3, a_3, b). $L_2^{-1} = 0.06$. $d = 0.0005$.

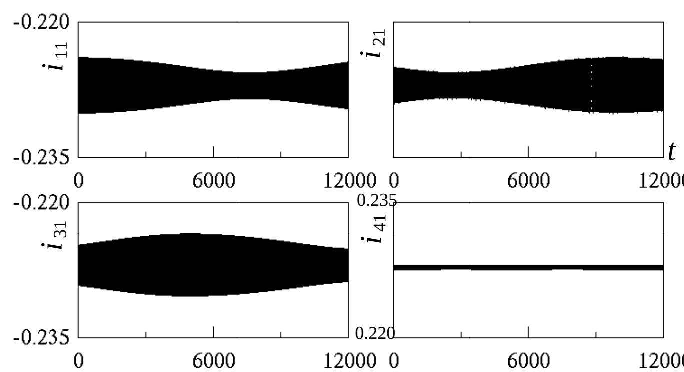


Figure 6: Waveforms. $L_2^{-1} = 0.04$. $d = -0.0015$.

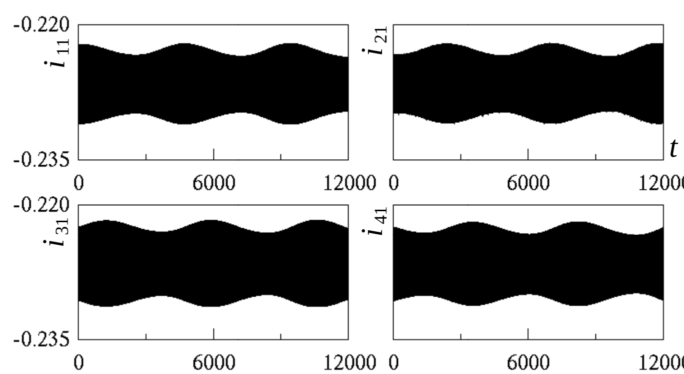
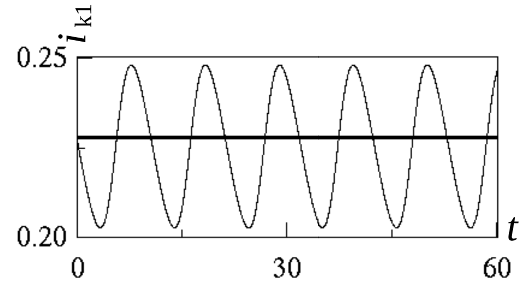
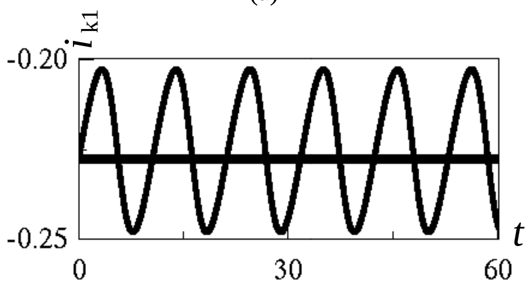


Figure 7: Waveforms. $L_2^{-1} = 0.06$. $d = -0.0021$.



(a)



(b)

Figure 8: Waveforms of oscillation death (unstable). $L_2^{-1} = 0.06$. $d = -0.0023$.

Acknowledgment

This work was partially supported by the Grant-in-Aid for Young Scientists (B) (No. 14780326) from the Ministry of Education of Japan.

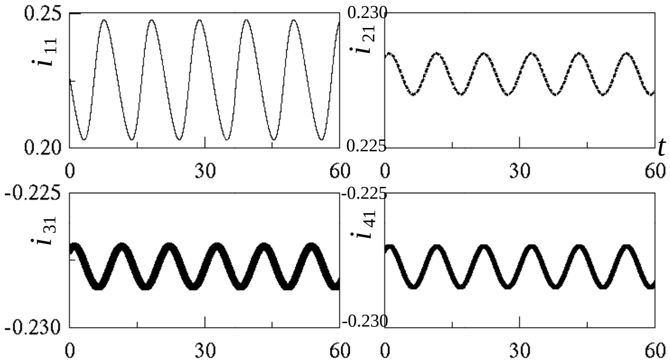


Figure 9: Waveforms of an interesting oscillatory mode (unstable). $L_2^{-1} = 0.06$. $d = -0.0021$.

References

[1] R.E. Kronauer, C.A. Czeisler, S.F. Pilato, M.C. Moore-Ede and E.D. Weitzman, “Mathematical model of the human circadian system with two interacting oscillators,” Am. J. Physiol. 242, pp.R3–R17, 1982.

[2] E.N. Brown, Y. Choe, H. Luithardt and C.A. Czeisler, “A statistical model of the human core-temperature circadian rhythm,” Amer. J. Physiol. Endocrinol. Metab., 279, E669–683, 2000.

[3] P.L. Buono and M. Golubitsky “Models of central pattern generators for quadruped locomotion,” *J. Math. Biol.* **42**, 291–326, 2001.

[4] T. Hirao, H. Nagashino and Y. Kinouchi, “Recruitment and Annihilation of the Oscillatory Activities in Coupled Neural Oscillators,” Proc. NOLTA’96, pp.5–8, Kouchi, Japan, Oct. 1996.

[5] M.De Sousa Vieira, A.J. Lichtenberg and M.A. Lieberman, “Self-synchronization of many coupled oscillators,” Int. J. Bifurcation and Chaos, vol.4, no.6, pp.1563–1577, 1994.

[6] A.T. Winfree, The geometry of biological time, Springer-Verlag, New York, 1980.

[7] M. Golubitsky, I. Stewart and D. G. Schaeffer, Singularities and Groups in Bifurcation Theory Vol. II, Springer-Verlag, New York, 1988.

[8] T. Endo and S. Mori, “Mode analysis of a ring of a large number of mutually coupled van der Pol oscillators,” IEEE Trans. Circuit & Syst., vol.CAS-25, no.1, pp.7–18, Jan. 1978.

[9] H. Sekiya, T. Yoshinaga, S. Moro, S. Mori and I. Sasase, “Investigation of Not-in-Phase Synchronization on Coupled System of Four Oscillators,” IEICE Trans. vol.J84-A, no.1, pp.42–52, Jan. 2001 (in Japanese).

[10] T. Mishima and H. Kawakami, “Bifurcation of equilibrium points and synchronized periodic solution in four coupled oscillators,” Proc. NOLTA’97, pp.37–40, 1997.

[11] K. Tsumoto, T. Yoshinaga and H. Kawakami, “Bifurcation of a Modified BVP Circuit Model for Neurons Generating Rectangular Waves,” IEICE Trans. Fundamentals, no.9, pp.1729–1736, Sept. 1999.

[12] S.V. Chakravarthy and J. Ghosh, “A complex-valued associative memory for storing patterns as oscillatory states,” *Biol. Cybern.*, **75**, pp.229–238, 1996.